Millimeter interferometers

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Millimeter interferometers

Outline

• The van Cittert–Zernike theorem
• The ideal interferometer
  ➔ geometrical delay, source size, bandwidth
• The real interferometer
  ➔ heterodyne receivers, delay correction, correlators
• Aperture synthesis
  ➔ uv plane, field of view, transfer function
• Sensitivity
van Cittert–Zernike theorem

- Wiener-Kichnine theorem (temporal)
  - temporal autocorrelation of \( S(t) = \text{FT(spectra)} \)
    \[ S(t_1) S(t_2) = \sum(\tau) \rightleftharpoons S(\nu) \]
  - implementation: FT spectrometers

- van Cittert–Zernike theorem (spatial)
  - spatial autocorrelation of \( S(x) = \text{FT(brightness)} \)
    \[ S(x_1) S(x_2) = \sum(u) \rightleftharpoons S(\alpha) \]
  - implementation: aperture synthesis
van Cittert–Zernike theorem

Young’s holes

Source

Interferences pattern

2D Fourier Transform

Measure with all possible holes distances

Visibility
van Cittert–Zernike theorem

Astronomical source

Interferences pattern

2D Fourier Transform

Measure with all possible baselines

Visibility
Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal
2. Do it for all possible scales
3. Take the FT and get an image of the brightness distribution
van Cittert–Zernike theorem

Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal \( \rightarrow \) 2-elements interferometer

2. Do it for all possible scales \( \rightarrow \) N antennas

3. Take the FT and get an image of the brightness distribution \( \rightarrow \) software
The ideal interferometer Sketch
The ideal interferometer

Measurements

- The heterodyne receiver measures the incoming electric field $E \cos(2\pi \nu t)$
- The correlator is a multiplier followed by a time integrator:
  \[
  r = \langle E_1 \cos(2\pi \nu t) \, E_2 \cos(2\pi \nu t) \rangle = E_1 E_2
  \]
- We have measured the spatial correlation of the signal!
- ...

...
The ideal interferometer

Measurements

- The heterodyne receiver measures the incoming electric field $E \cos(2\pi \nu t)$
- The correlator is a multiplier followed by a time integrator:
  \[
  r = \langle E_1 \cos(2\pi \nu t) \ E_2 \cos(2\pi \nu t) \rangle = E_1 \ E_2
  \]
- We have measured the spatial correlation of the signal!
- But we have forgotten the geometrical delay
The ideal interferometer

Sketch

\[ \tau_g = b \cdot s / c \]
The ideal interferometer
Measurements

- There is a geometrical delay $\tau_g$ between the two antennas → more complex experiment than the Young’s holes
- Correlator output:

$$r = \langle E_1 \cos(2\pi \nu t) \ E_2 \cos(2\pi \nu t) \rangle = E_1 E_2$$

$$r = \langle E_1 \cos(2\pi \nu (t - \tau_g)) \ E_2 \cos(2\pi \nu t) \rangle$$

$$= E_1 E_2 \cos(2\pi \nu \tau_g)$$
The ideal interferometer
Measurements

- Correlator output: \( r = E_1 E_2 \cos(2\pi \nu \tau_g) \)
- \( \tau_g \) varies slowly with time (Earth rotation) \( \longrightarrow \) fringes
- Natural fringe rate:

\[
\tau_g = \frac{b \cdot s}{c} \quad \nu \frac{d\tau_g}{dt} \approx \Omega_{\text{earth}} \frac{b \nu}{c}
\]

\( \sim 50 \text{ Hz} \) for \( b = 800 \text{ m} \) and \( \nu = 250 \text{ GHz} \)
The ideal interferometer
Measurements

- Correlator output: \( r = E_1 E_2 \cos(2\pi \nu \tau_g) \)
- \( \tau_g \) varies slowly with time (Earth rotation) \( \longrightarrow \) fringes
- \( \tau_g \) is known from the antenna position, source direction, time \( \longrightarrow \) could be corrected
- Problems: the source is not a point source
  the signal is not monochromatic
The ideal interferometer

Source size

\[ s = s_0 + \sigma \]

Power received from \( d\Omega \):
\[ A(s)I(s)d\Omega \]
\[ A(s) = \text{beam} \]
\[ I(s) = \text{source} \]

Correlator output:
\[ r = E_1 E_2 \cos(2\pi \nu \tau_g) \]
\[ r = A(s)I(s)d\Omega \cos(2\pi \nu \tau_g(s)) \]
The ideal interferometer
Source size

- Correlator output integrated over source:

\[
R = \int_{Sky} A(s)I(s) \cos(2\pi \nu b \cdot s / c) \, d\Omega \\
= |V| \cos(2\pi \nu \tau_g - \varphi_V)
\]

- Complex visibility:

\[
V = |V|e^{i\varphi_V} = \int_{Sky} A(\sigma)I(\sigma)e^{-2i\pi \nu b \cdot \sigma / c} \, d\Omega
\]
The ideal interferometer

Source size

\[ R = \int_{Sky} A(s)I(s) \cos(2\pi \nu b \cdot s/c) d\Omega \]

\[ = \cos \left( 2\pi \nu \frac{b \cdot s_0}{c} \right) \int_{Sky} A(\sigma)I(\sigma) \cos(2\pi \nu b \cdot \sigma/c) d\Omega \]

\[ - \sin \left( 2\pi \nu \frac{b \cdot s_0}{c} \right) \int_{Sky} A(\sigma)I(\sigma) \sin(2\pi \nu b \cdot \sigma/c) d\Omega \]

\[ = \cos \left( 2\pi \nu \frac{b \cdot s_0}{c} \right) |V| \cos \varphi_V - \sin \left( 2\pi \nu \frac{b \cdot s_0}{c} \right) |V| \sin \varphi_V \]

\[ = |V| \cos(2\pi \nu \tau_g - \varphi_V) \]
The ideal interferometer

Summary

• Correlator output:

\[ r = \langle E_1 \cos(2\pi \nu t) \ E_2 \cos(2\pi \nu t) \rangle = E_1 E_2 \]
\[ r = E_1 E_2 \cos(2\pi \nu \tau_g) \quad \leftarrow \text{delay} \]
\[ R = |V| \cos(2\pi \nu \tau_g - \varphi_V) \quad \leftarrow \text{source size} \]

• Complex visibility \( V \) resembles a Fourier Transform:

\[ V = |V|e^{i\varphi_V} = \int_{Sky} A(\sigma)I(\sigma)e^{-2i\pi \nu \mathbf{b} \cdot \sigma/c} d\Omega \]
The ideal interferometer

Summary

- **Correlator output:**

  \[ r = \langle E_1 \cos(2\pi \nu t) E_2 \cos(2\pi \nu t) \rangle = E_1 E_2 \]

  \[ r = E_1 E_2 \cos(2\pi \nu \tau_g) \quad \text{← delay} \]

  \[ R = |V| \cos(2\pi \nu \tau_g - \varphi_V) \quad \text{← source size} \]

- **3D version of van Cittert–Zernike**

  - We do **not** measure \( r = FT(I) \)

  - We measure \( R = \text{something related to } V, \text{ which resembles the } FT(I) \)
The ideal interferometer

Bandwidth

- Integrating over a finite bandwidth $\Delta \nu$

$$R = \frac{1}{\Delta \nu} \int_{\nu_0-\Delta \nu/2}^{\nu_0+\Delta \nu/2} |V| \cos(2\pi \nu \tau_g - \varphi_V) \, d\nu$$

$$= |V| \cos(2\pi \nu_0 \tau_g - \varphi_V) \frac{\sin(\pi \Delta \nu \tau_g)}{\pi \Delta \nu \tau_g}$$

- The fringe visibility is attenuated by a $\sin(x)/x$ envelope (=$\text{bandwidth pattern}$) which falls off rapidly
The ideal interferometer

Summary

- Correlator output:

\[ r = \langle E_1 \cos(2\pi \nu t) E_2 \cos(2\pi \nu t) \rangle = E_1 E_2 \]

\[ r = E_1 E_2 \cos(2\pi \nu \tau_g) \]

\[ R = |V| \cos(2\pi \nu \tau_g - \varphi_V) \]

\[ R = |V| \cos(2\pi \nu_0 \tau_g - \varphi_V) \frac{\sin(\pi \Delta \nu \tau_g)}{\pi \Delta \nu \tau_g} \]

- We measure \( R \), which is related to \( V \), which resembles the FT(\( I \)). \( R \) also depends on \( \tau_g \).
The ideal interferometer
Delay correction

\[ R = |V| \cos(2\pi\nu_0\tau_g - \varphi_V) \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g} \]

- \( \tau_g \) varies with time because of the Earth rotation \( \rightarrow \) rapid decrease of \( R \) (1\% for a path length difference of \( \sim 2 \text{ cm} \) and \( \Delta\nu = 1\text{GHz} \))
- Tracking a source requires the **compensation of the geometrical delay**
- Interferometry requires temporal coherence!
The ideal interferometer

Delay correction

\[ R = |V| \cos(2\pi \nu_0 \tau_g - \varphi_V) \frac{\sin(\pi \Delta \nu \tau_g)}{\pi \Delta \nu \tau_g} \]

- Tracking a source requires the **compensation of the geometrical delay**
- This can be achieved by introducing an **instrumental delay** in the correlator
- If delay is compensated, one can measure \( R = |V| \cos(\varphi_V) \)
The real interferometer Sketch

\[ \tau_g = b \cdot s/c \]
The real interferometer
Heterodyne detection

- In the receiver **mixer**, the incident electric field is combined with a **local oscillator** signal

\[ U(t) = E \cos (2\pi\nu t + \phi) \]
\[ U_{LO}(t) = E_{LO} \cos (2\pi\nu_{LO} t + \phi_{LO}) \]
\[ \nu_{LO} \approx \nu \]

- The mixer is a **non-linear** element:

\[ I(t) = a_0 + a_1(U + U_{LO}) + a_2(U + U_{LO})^2 + a_3(...)^3 + ... \]
The real interferometer
Heterodyne detection

- There are terms at various frequencies and harmonics
- A **filter** selects the frequencies such that:

  \[ \nu_{IF} - \Delta \nu / 2 \leq |\nu - \nu_{LO}| \leq \nu_{IF} + \Delta \nu / 2 \]

- \( \nu_{IF} \) is the **intermediate frequency**
- \( \nu_{IF} \) such that amplifiers and transport elements available
- PdBI: \( \nu_{IF} = 4-8 \) GHz, ALMA: \( \nu_{IF} = 4-12 \) GHz
The real interferometer

Heterodyne detection

- The receiver output is

\[ I(t) \propto E E_{LO} \cos \left( \pm \left( \pm 2\pi (\nu - \nu_{LO}) t + \varphi - \varphi_{LO} \right) \right) \]
The real interferometer

Heterodyne detection

- **DSB** receivers accept both LSB and USB frequencies, i.e. their output is the sum of LSB and USB
- **SSB** receivers accept only LSB or USB (response very strongly frequency dependant)
- **2SB** receivers are 2 DSB receivers combined such that the two bands are independently output (and processed)
The real interferometer
Delay tracking

- A compensating delay is introduced in one of the branch of the interferometer, on the IF signal
- Equivalent to the delay lines in IR interferometers
The real interferometer
Delay tracking

- Phases of the two signals (USB):
  \[ \varphi_1 = 2\pi \nu \tau_g \quad \varphi_1 = 2\pi \nu \tau_g = 2\pi (\nu_{\text{LO}} + \nu_{\text{IF}}) \tau_g \]
  \[ \varphi_2 = 0 \quad \varphi_2 = 2\pi \nu_{\text{IF}} \tau_i \]

- Correlator output:
  \[ R = |V| \cos(2\pi \nu \tau_g - \varphi_V) \]
  \[ R = |V| \cos(\varphi_1 - \varphi_2 - \varphi_V) \]
  \[ R = |V| \cos(2\pi \nu_{\text{LO}} \tau_g - \varphi_V) \]
Delay tracking not enough because applied on the IF

Solution: in addition to delay tracking, rotate the phase of the local oscillator such that at any time:

$$\varphi_{LO}(t) = 2\pi \nu_{LO} \tau_g(t)$$

$$\tau_g$$ is computed for a reference position = phase center

Phase center = pointing center in practice, though not mandatory
The real interferometer
Fringe stopping

- Phases of the two signals (USB):
  \[ \varphi_1 = 2\pi \nu \tau_g = 2\pi (\nu_{LO} + \nu_{IF}) \tau_g \]
  \[ \varphi_2 = 2\pi \nu_{IF} \tau_i + \varphi_{LO} \]
  \[ \varphi_{LO} = 2\pi \nu_{LO} \tau_g \]

- Correlator output:
  \[ R = |V| \cos(\varphi_1 - \varphi_2 - \varphi_V) \]
  \[ R = |V| \cos(\varphi_V) \]
The real interferometer
Complex correlator

• After fringe stopping:

\[ R = |V| \cos(-\varphi_V) \]

• The corrections were so good that there is **no time or delay dependance** any more \(\longrightarrow\) cannot measure \(|V|\) and \(\varphi_V\) separately.

• A second correlator is necessary, with one signal phase shifted by \(\pi/2\):

\[ R_i = |V| \sin(-\varphi_V) \]

• **The complex correlator measures directly the visibility**
The real interferometer
Complex correlator

- The correlator measures the real and imaginary parts of the visibility. **Amplitude and phases are computed off-line.**
- Amplitude and phases have more physical sense
  - Visibility amplitude = correlated flux
  - The atmosphere adds a **phase** to the incoming signals
    \[ \text{measured phase} = \text{visibility} + \varphi_1 - \varphi_2 \]
• Remember the Wiener-Kichnine theorem?

• Calculate the correlation function for several delay $\delta \tau \rightarrow$ measurement of the temporal correlation $\rightarrow$ FT to get the spectra:

$$V_\nu(u, v, \nu) = \int V(u, v, \tau)e^{-2i\pi\nu \delta \tau}d\nu$$

• Nothing to do with geometrical delay compensation – $\delta \tau \sim 1/\delta \nu$ here

• Mixed up implementation in correlator software
Bas. 13 L04 L08 LSB

Real vs. Sky Frequency

8.863 $10^4$ 8.8635 $10^4$

Bas. 13 L04 L08 LSB

Imaginary vs. Sky Frequency

8.863 $10^4$ 8.8635 $10^4$

Bas. 13 L04 L08 LSB

Amplitude (K) vs. Sky Frequency

8.863 $10^4$ 8.8635 $10^4$

Bas. 13 L04 L08 LSB

Phase vs. Sky Frequency

8.863 $10^4$ 8.8635 $10^4$
Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal → 2-elements interferometer
2. Do it for all possible scales → N antennas
3. Take the FT and get an image of the brightness distribution → software
Aperture synthesis
Complex visibility

- Complex visibility:

\[ V = |V| e^{i\varphi_V} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi \nu b \cdot \sigma / c} d\Omega \]

- Going from 3-D to 2-D? ...some algebra...

- OK providing that:

\[
(max. \text{ field of view})^2 \times \text{max. baseline} \ll 1
\]

\[
\Rightarrow \frac{(max. \text{ field of view})^2}{\text{resolution}} \ll 1
\]
Aperture synthesis
Complex visibility

\[ V(u, v) = \int_{Sky} A(\ell, m)I(\ell, m)e^{-2i\pi v(ul+vm)}d\Omega \]

• \(uv\) plane is perpendicular to the source direction, **fixed wrt source** \(\rightarrow\) back to Young’s hole

• Price: limit on the field of view

• Approximation **ok in (sub)mm domain**, problem at wavelengths \(>\) cm
Aperture synthesis
(Field of view)

- Field of view is limited by
  - the **antenna primary beam**: the interferometer measures $A \times I$
  - the **2D visibility approximation**
  - the frequency averaging (bandwidth)
  - the time averaging (integration)
    $\leftrightarrow$ averaging in the $uv$ plane; possible only if limited field of view
Aperture synthesis
(Field of view)

- Values for Plateau de Bure

<table>
<thead>
<tr>
<th>θ_s (arcsec)</th>
<th>ν (GHz)</th>
<th>2-D Field (arcmin)</th>
<th>0.5 GHz Bandwidth (arcsec)</th>
<th>1 Min Averaging (arcmin)</th>
<th>Primary Beam (arcsec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5''</td>
<td>80</td>
<td>5'</td>
<td>80''</td>
<td>2'</td>
<td>60''</td>
</tr>
<tr>
<td>2''</td>
<td>80</td>
<td>3.5'</td>
<td>30''</td>
<td>45''</td>
<td>60''</td>
</tr>
<tr>
<td>2''</td>
<td>230</td>
<td>3.5'</td>
<td>1.5'</td>
<td>45''</td>
<td>24''</td>
</tr>
<tr>
<td>0.5''</td>
<td>230</td>
<td>1.7'</td>
<td>22''</td>
<td>12''</td>
<td>24''</td>
</tr>
</tbody>
</table>

- Problem with 2D field: software; with bandwidth: split the data for imaging; with time averaging: dump faster.

- Primary beam is the main limit on the FOV
Aperture synthesis
Complex visibility

\[ V(u, v) = \int_{Sky} A(\ell, m)I(\ell, m)e^{-2i\pi\nu(\ell u + m v)}d\Omega \]

- \(uv\) plane is perpendicular to the source direction, fixed wrt source \(\rightarrow\) back to Young's hole
- Price: limit on the field of view
- Approximation **ok in (sub)mm domain**, problem at wavelengths > cm
Aperture synthesis

\textit{uv plane}

- \textit{uv} plane is perpendicular to the source direction, \textbf{fixed wrt source} \text{ back to Young’s hole}

- \((u, v)\) is the 2–antennas \textbf{vector} baseline projected on the plane perpendicular to the source

- \((u, v)\) are \textbf{spatial frequencies}

- ... Earth rotation ... \textit{ (spherical trigonometry) ...}

- \((u, v)\) describe an \textbf{ellipse} in the \textit{uv} plane (for \(\delta = 0\) deg, a line)
Aperture synthesis
uv plane coverage
• We started with Young’s hole experiment and the van Cittert–Zernike theorem
• An interferometer is more complex, because the two antennas (holes) are not in a plane perpendicular to the source direction \( \rightarrow \) geometrical delay, etc.
• What we are measuring is not \( \text{FT}(I) \), but the visibility \( V \), which resembles a FT
• For small field of view = practical case, \( V \) is the 2D FT of the sky brightness distribution (\( \times \) the primary beam)
• Back to the van Cittert–Zernike theorem
Aperture synthesis
Image formation

Measurements = uv plane sampling × visibilities
After FT: dirty map = dirty beam * image

The FT of the $uv$ plane coverage gives the dirty beam = the PSF of the observations
Aperture synthesis

Image formation

Max. baseline gives the angular resolution
Aperture synthesis
Image formation

Single-dish observations of a point source

Aperture function $\iff$ Voltage pattern $\Rightarrow |\cdot|^2$

Power pattern

$= \text{PSF} = \text{Primary beam}$
Aperture synthesis
Image formation

Single-dish observations of a point source

Aperture function $\Rightarrow$ Voltage pattern

\[ \star \downarrow \]

Transfer function $T(u, v)$ $\Rightarrow$ Power pattern

\[ \downarrow |.|^2 \]

$= \text{PSF} = \text{Primary beam}$

Transfer function describes how spatial frequencies are transmitted by the telescope
Aperture synthesis

Image formation

Interferometer observation of a point-source

Aperture function \[ \star \]

Transfer function \( T(u, v) \) \[ \Rightarrow \]

Voltage pattern \[ \downarrow | \cdot |^2 \]

Power pattern \( B(\ell, m) \)

= PSF

= synthesized beam

Aperture synthesis = sample directly the transfer function of a huge instrument (diameter \( \sim \) max. baseline).

Aperture function? Does not exist...
Measurement of visibilities is limited by noise emitted by atmosphere, antenna, ground, receivers.

The rms noise for the baseline $ij$ is given by:

$$
\delta S_{ij} = \frac{2k}{A\eta_A\eta_Q\eta_P} \cdot \frac{\sqrt{T_{SYSi}T_{SYSj}}}{\sqrt{2BT}}
$$

- $A$ antenna physical aperture
- $\eta_A$ antenna aperture efficiency
- $\eta_Q$ efficiency for the correlator
- $T_{SYSi}$ system noise temperature (single dish)
- $B$ bandwidth
- $T$ integration time
- $\eta_P$ phase decorrelation factor (LO jitter)
• This is the noise on the real and on the imaginary parts of the visibilities (measured independently)
• This is also the noise on the amplitude $S$
• Noise on the phase more complex, of the order of $\sigma / S$
For $N$ identical antenna/receivers, i.e. $N(N - 1)/2$ baselines, the point-source sensitivity is:

$$\delta S = \frac{2k}{A\eta_A\eta_Q\eta_P} \cdot \frac{T_{\text{SYS}}}{\sqrt{N(N - 1)BT}}$$

- Scales as $\sim 1/N$
- Sensitivity to extended sources depends on angular resolution
Sensitivity
Phase decorrelation

- Short term phase errors in the local oscillators (jitter) will cause a decorrelation of the signal and reduce the visibility amplitude by a factor

\[ \eta_P(\text{baseline12}) = e^{-\left(\sigma_1^2 + \sigma_2^2\right)/2} = \eta_1 \eta_2 \]

- Requirements:

<table>
<thead>
<tr>
<th>( \eta_1 )</th>
<th>0.99</th>
<th>0.98</th>
<th>0.95</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 ) (degrees)</td>
<td>8.1</td>
<td>11.5</td>
<td>18.3</td>
<td>26.4</td>
</tr>
</tbody>
</table>
Sensitivity

Phase decorrelation

• $\eta_P = 0.9 \rightarrow \eta_1 = 0.95 \rightarrow \sigma_1 = 18 \text{ deg}$

• PdBI: LO derived from a reference at 1.8 GHz

• Phase stability required $= \sigma_1 (1.8 \text{ GHz}/230 \text{ GHz}) \sim 0.15^\circ$

• Very stable oscillators are required

• Phase decorrelation due to the atmosphere is a more severe problem
Sensitivity

Phase decorrelation

- $\eta_P = 0.9 \rightarrow \eta_1 = 0.95 \rightarrow \sigma_1 = 18 \text{ deg}$
- PdBI: LO derived from a reference at 14 GHz
- Phase stability required = $\sigma_1 \ (14 \text{ GHz}/230 \text{ GHz}) \sim 1^\circ$
- **Very stable** oscillators are required
- Phase decorrelation due to the atmosphere is a more severe problem
Summary
Other instrumental issues

- Phase lock systems to control $\varphi_{LO}$
- Real-time monitoring and correction of the phase offset in the cables or fibers
- Complex phase switching is used to cancel offsets, separate/reject side bands, ...
- Antenna position measurements, to get the delay, u, v
- Antenna deformations, e.g. thermal expansion (delay)
- Accurate focus measurements (delay)
- Atmospheric phase monitoring
- ...
Summary

It works!