Study of NMR maser instability in hyperpolarized liquid $^3$He

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Abstract

We present studies of noise-triggered NMR masers, and discuss the existing descriptions of this phenomenon. We then develop our own model and test it experimentally, thanks to dedicated NMR measurements and careful data analysis. The developed model is shown to be on good tracks but still needs improvement: it must take into account additional non-linear NMR effects like distant dipolar fields and non-exponential transverse relaxation, which were neglected in the first approach.

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Introduction

The purpose of my internship was the study of noise-triggered NMR masers in $^3$He - $^4$He mixtures. It involved a theoretical part to describe the noise-triggering phenomenon, and an experimental part to obtain and exploit data for a large number of such masers.

The internship also included a study of the compensation of eddy-currents created by the switching of gradients in an NMR setup. This smaller part of my work will not be discussed in this report.

Nuclear Magnetic Resonance

The usual vector model of NMR provides a simple representation of the behaviour of an isolated spin 1/2 (or of a bulk magnetization without internal interactions). The magnetization is represented as a vector evolving in the so-called Bloch sphere, with its equilibrium position along the $+z$-axis. It corresponds to the ground-state of a 2-level quantum system, the excited state corresponding to the diametrically opposed point, along the $-z$ axis.

The evolution of the magnetization vector obeys the Bloch equations:

\[
\begin{align*}
\frac{dM_x(t)}{dt} &= \gamma (\vec{M}(t) \times \vec{B}(t))_x - \frac{M_x(t)}{T_2^\ast}
\frac{dM_y(t)}{dt} &= \gamma (\vec{M}(t) \times \vec{B}(t))_y - \frac{M_y(t)}{T_2^\ast}
\frac{dM_z(t)}{dt} &= \gamma (\vec{M}(t) \times \vec{B}(t))_z - \frac{M_z(t)-M_0}{T_1}
\end{align*}
\]

where $\vec{B}$ is the magnetic field and $\gamma$ the gyromagnetic ratio. $1/T_1$ is the longitudinal relaxation rate, and corresponds to the linewidth of the transition for the 2-level quantum system. $1/T_2$ is the intrinsic transverse relaxation rate, corresponding to the decoherence rate for the quantum system coupled to a reservoir. When the total magnetization is considered, $1/T_2^\ast$ is replaced by $1/T_2^\ast$, an effective rate that takes into account the transverse loss of phase coherence due to field inhomogeneities.

Since we will not consider any spin-exchange phenomena, we will keep this phenomenological, classical description of NMR from now on.

Initially, NMR used the magnetization present at thermal equilibrium in a magnetic field, using higher and higher fields to increase the magnetization (and therefore the recorded signal).

More recently, NMR of hyperpolarized species has been developed. It implies the polarization of the studied species far above the thermal equilibrium level, usually by means of optical pumping. The high levels of magnetization achieved through this technique revealed non-linear effects that were negligible for low-magnetization NMR.
Nonlinear NMR effects

Non-linearity in NMR arises when the fields created by the sample itself become non-negligible in comparison with other causes of evolution (in the Bloch equation). Therefore, low-field and high-magnetization experiments such as the ones carried out with hyperpolarized samples are a perfect test-bed to reveal these collective effects [1, 2].

The first non-linear effect is the dipolar field. It corresponds to the field created by each spin and felt by its neighbours. In liquid-state NMR, this interaction is expected to average out, since Brownian motion allows neighbouring spins to probe a $4\pi$ space around each other. However, this is only true for close neighbours. On larger distances, diffusion in too slow to allow the dipolar fields to average out, therefore distant dipolar fields can play a role in the evolution of the magnetization [2].

A first experimental effect of distant dipolar fields is spectral clustering, observed when the spectrum of the NMR signal of a magnetization evolving in an inhomogeneous field shows several unexpectedly sharp lines. It corresponds to the appearance of collective magnetization modes in the sample, which evolve in an independent and very stable way [1, 3].

Another effect of dipolar fields is the dynamical instabilities that can lead to the abrupt decay of NMR signal [1, 4, 5], due to the growth of little deviations throughout the sample in an imperfectly homogeneous field.

The other major non-linear effect arises from the interaction of the magnetization with the detection coil. The precessing magnetization induces a current in the pick-up coil, which itself produces a RF frequency field perfectly on resonance with the Larmor frequency of the spins. This field tends to nutate the magnetization back to the $+z$-axis [1, 6, 2]. This process is known as radiation damping.

The result of radiation damping is the collective flipping of the magnetization in the sample, resulting in a maser burst.

Maser emissions

For maser emission to occur, the radiation damping rate $1/\tau_R$ (which tends to flip the magnetization, and therefore create transverse magnetization starting from the $-z$-axis) must be stronger than the transverse relaxation rate $1/T_2$ (which tends to destroy the global transverse magnetization created by radiation damping). The radiation damping rate depends on the total magnetization as well as on the coupling between the sample and the pick-up coil. From a quantum point of view, the sample above maser threshold can also be seen as an inverted population of 2-level systems strongly coupled to a resonant cavity (hence the maser denomination).
A maser starting from an initial state close to the $-z$-axis is expected to return to the stable position along the $+z$-axis \[2,7\]. However, in the presence of distant dipolar fields, the precession is unstable, and the shape of the maser signal is not symmetrical with respect to the $xy$ plane any more; chaotic multiple maser emissions can occur \[7\] (fig. 4).

It is important to point out that maser emissions arise from the instability of the magnetization position along the $-z$-axis, the maser can therefore be triggered by any excitation (be it only noise) that will push the magnetization off this unstable equilibrium.

In this document, we will first report the existing theoretical approaches to the problem of noise-triggered NMR masers and compare it with our own. Then, we will describe the experimental process that provided the data to be compared with this theory, and the details of the data analysis method used to extract the needed information from the raw data. Finally, we will discuss the results, assess their quality and give the prospects of this work.

1 Theoretical approach

1.1 Existing descriptions

1.1.1 Sodickson et al., 1996

In their 1996 paper \[6\], A. Sodickson et al. provided a first look at the maser triggering by noise. Their description of the maser offers a clear physical picture of the phenomenon. They derived the dynamics of the magnetization through modified Bloch equations, for a maser without distant dipolar fields in the case of a small initial angle $\theta$ between the magnetization and the $-z$ axis.

In the case of a system with magnetization perfectly along $-z$, they noticed that maser emission still occurs, since this configuration is unstable and is pushed out of equilibrium by noise (they suggested spectrometer feed-through or thermal noise in the coil as sources of noise). The authors provided a qualitative description of the way the noise creates the small initial angle needed to trigger the maser. Yet, the article does not link quantitatively the statistics of the noise with those of the observed noise-triggered masers.

1.1.2 Augustine et al., 2000

Augustine et al. gave another description of the phenomenon in 2000 \[8\]. Their approach used mostly the tools of statistical physics, describing the noise triggering with a Langevin equation, and working out the statistics with such tools as equipartition and correlation functions.

The computations, although detailed, are not fully convincing on several points, especially in the way the transition from the equilibrium (along $+z$)
to non-equilibrium (along $-z$ and during maser emission) is tackled.

Besides, the presented experimental results show surprising discrepancies with the theory. For instance, the moment at which the maser brings the magnetization in the $xy$ plane, $T_{\text{delay}}$, agreed within a few percent with the predicted value (although the distribution seems more peaked than expected), while the average spread of the initial angle due to noise $\langle \theta^2 \rangle_\infty$ differs from the prediction by up to 3 orders of magnitude.

1.2 Our approach

Both of the previous studies were carried out in conventional high-field NMR spectrometers, with a magnetization low enough to have negligible distant dipolar fields. Their experimental setup allowed the authors to measure fully symmetric maser emissions with a simple return of the magnetization along the $+z$ axis [6, 7]. The studied parameter of the emission was therefore $T_{\text{delay}}$. In our case, the study requires to focus only on the beginning of each maser emission (see 2.3.2), so that the statistics are rather focused on the (fictitious) starting angle of the maser $\theta_0$ (see 1.3).

Like Sodickson et al., we chose a dynamical approach, based on the Bloch equations, modified in order to include the radiation damping, but we also added the noise as an unspecified time-dependent field. This allows to derive the noise-dependent dynamics, and therefore explicitly link the noise statistics with the statistics of the maser starting angle.

A key objective of the calculation is to get all prefactors right, so as to be able to make quantitative comparisons with the experiments. As these prefactors are often left aside or are convention-dependent, no usual tool such as Rotating Wave Approximation is taken for granted, and the prefactors are carefully re-defined (e.g. for the Fourier Transformed noise field) and kept self-consistent.

1.3 Calculations

The details of the calculations can be found in the Appendix.

1.3.1 Definition of the problem, Bloch equations

We assume without loss of generality that the noise-induced magnetic field is along the pick-up coil’s axis (here, the x-axis) only.

We consider the small angle limit, where the magnetization is almost exactly along the $-z$ axis, so that $M_z = \text{cst} = M_0$. In the rotating frame (at Larmor frequency $\Omega_0$), the time evolution of the complex transverse
magnetization ($M_T = M_x + iM_y$) follows:
\[
\frac{\partial}{\partial t} M_T = \frac{M_T}{T_R} + \gamma M_0 \left( \sin(\Omega_0 t) B_n(t) + i \cos(\Omega_0 t) B_n(t) \right)
\]
\[
= \frac{M_T}{T_R} + i\gamma M_0 \exp(-i\Omega_0 t) B_n(t)
\]

where $B_n(t)$ is the magnetic (noise) field along the x-axis, and $T_R = \frac{T^*_2 - \tau_R}{T^*_2}$ is the effective growth time of the radiation damping (where $1/\tau_R$ is the rate of radiation damping alone, and $1/T^*_2$ is the NMR transverse relaxation rate) - to be compared with [6].

We assume $M_T(0) = 0$ for a perfect $\pi$-pulse. $M_T(t)$ can be formally integrated to:
\[
M_T(t) = i\gamma M_0 e^{t/T_R} \cdot \int_0^t B_n(t') e^{-i\Omega_0 t'} e^{-t'/T_R} dt'
\]
\[
= i\gamma M_0 e^{t/T_R} K(t)
\]

The (noisy) integrand of $K(t)$ is exponentially damped over time, so that $K(t)$ is expected to converge towards a finite value $K_0$. In this limit, we have $M_T(t) \sim i\gamma M_0 K_0 e^{t/T_R}$.

### 1.3.2 Noise statistics

We define the Fourier Transform of the noisy field, $\tilde{b}(\omega)$:
\[
B_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \tilde{b}(\omega) d\omega
\]

and rewrite $K(t)$ as
\[
K(t) = \frac{1}{2\pi} \int_0^t dt' \int_{-\infty}^{+\infty} d\omega \tilde{b}(\omega) e^{-i\omega t'} e^{-i\Omega_0 t'} e^{-t'/T_R}
\]

Integration over time leads to:
\[
K(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{b}(\omega) e^{-t/T_R} e^{i\Psi(\omega)} \sqrt{1 + e^{2t/T_R} - 2 \cos((\Omega_0 + \omega)t)e^{t/T_R}} d\omega
\]

where $\Psi(\omega)$ is the phase of the time integral.

For a white noise, we expect $\tilde{b}(\omega)$ to be a flat distribution with random phases, such that $\langle \tilde{b}(\omega) \rangle = 0$ and $\langle \tilde{b}(\omega) \tilde{b}(\omega')^* \rangle = \alpha^2 \delta(\omega - \omega')$. Actually, the white noise is filtered by the pick-up circuit, which results in a bell-shaped spectral distribution $\alpha^2(\omega)$, centered on $\Omega_0$ and with a bandwidth $\sim \Omega_0/Q$. 

8
$K(t)$ behaves as expected at long times, converging towards a finite (complex) value: $K_0 = \frac{1}{2\pi} \int \hat{b}(\omega) e^{i\Psi(\omega)} d\omega \sqrt{(\Omega_0 + \omega)^2 + 1/T_R}$. We also obtain that $\langle K_0 \rangle = 0$.

$$\langle |K_0|^2 \rangle = \frac{1}{4\pi^2} \int \int \frac{\langle \hat{b}(\omega) \hat{b}(\omega')^* e^{i(\Psi(\omega) - \Psi(\omega'))} \rangle d\omega d\omega'}{\sqrt{(\Omega_0 + \omega)^2 + 1/T_R^2} \sqrt{(\Omega_0 + \omega')^2 + 1/T_R^2}}$$

$$= \frac{1}{4\pi^2} \int \int \frac{\alpha^2(\omega) \delta(\omega - \omega') e^{i(\Psi(\omega) - \Psi(\omega'))} d\omega d\omega'}{\sqrt{(\Omega_0 + \omega)^2 + 1/T_R^2} \sqrt{(\Omega_0 + \omega')^2 + 1/T_R^2}}$$

$$= \frac{1}{4\pi^2} \int \frac{\alpha^2(\omega) d\omega}{(\Omega_0 + \omega)^2 + 1/T_R^2}$$

Since $1/T_R$ is of the order of 1Hz $\ll \Omega_0/Q$, we can assume $\alpha$ to be constant over the integration, and we get:

$$\langle |K_0|^2 \rangle \approx \frac{\alpha^2(\Omega_0)}{4\pi} T_R$$

### 1.3.3 Statistics of maser triggering

We define $x_0 = -i\gamma K_0$, so that for $t \gg T_R$, $M_T(t) \sim M_0 x_0 e^{i/T_R}$. If we represent the motion of the magnetization in the Bloch sphere, for $t \gg T_R$ the evolution of the maser becomes similar to that of a maser triggered by a small tipping angle between $\vec{M}$ and $-z$, such as described in [6]. We write $x_0 = \theta_0 e^{i\phi_0}$. $\theta_0$ corresponds to the starting angle of this equivalent maser.

Based on the central limit theorem, we assume that the white noise induces symmetric independent Gaussian distributions for $M_x$ and $M_y$. Therefore $x_0$ follows a two-dimensional Gaussian distribution.

The expected distribution for the radial part of a 2D Gaussian distribution is given by:

$$P(\theta_0) = \frac{\pi}{2(\theta_0)^2} \theta_0 e^{-\frac{\pi}{2(\theta_0)^2}}$$

with $\langle \theta_0^2 \rangle = \frac{\pi}{4} \langle \theta_0 \rangle^2$

Therefore:

$$\langle \theta_0 \rangle = \frac{\gamma \alpha(\Omega_0)}{4\pi} \sqrt{T_R}$$

### 1.4 Discussion

Our derivation only focuses on the small angle limit, resulting in an exponential growth of the maser. Once the maser has been triggered, we have shown that the noise become negligible. The derivation of the large angle behaviour (that can be found in [7] and references therein) is therefore still valid in
our case. It must be stressed though that all these derivations assume that the field inhomogeneities keep the FID exponential (through the use of $T_2^*$), and do not take into account dipolar fields.

2 Experiments

The theory gave us an expected statistical distribution of maser starting angles $\theta_0$ depending on noise power. The goal of the experiments will be to record a large number of maser starts, and to measure as precisely as possible $\theta_0, T_R$ and $\alpha(\Omega_0)$, in order to compare these statistics with the prediction.

2.1 Experimental setup

Most of the elements of the experimental setup described below can be found with more details in [9].

2.1.1 NMR setup

The NMR setup works at cryogenic conditions with a very low magnetic field. The main field is created by a set of 7 circular coils, and shimmed with gradient coils along the 3 axes (fig. 1 left). The total static field has a value of 2,285 mT, corresponding to a Larmor frequency for $^3$He of 74,1 kHz.

![Figure 1](image.png)

Figure 1: Schematics of the NMR hardware : main field coils and shimming gradients outside the cryostat; longitudinal gradient, excitation and pick-up coils around the sample inside it (adapted from [9]).

The sample cell is contained in a vacuum tank, itself enclosed in helium cryostat at 4 K. The vacuum tank contains a $^4$He pot, connected to the liquid helium tank. Pumping the helium pot allows the temperature to go as low as 1,1 K. The sample cell is surrounded by a sealed volume of superfluid helium, which ensures a uniform thermal contact with the cold $^4$He pot to which it
The sample cell is connected to the polarization cell, in which hyperpolarization is achieved through Metastability Exchange Optical Pumping (MEOP) at room temperature (see [9] for details). The polarized $^3$He is flushed towards the sample cell by a flux of $^4$He and is dissolved in the liquid $^4$He that it contains. The excitation coil ($B_1$) and the pick-up coil, as well as a $z$-gradient, surround this double-walled sample volume (fig. 1, right).

### 2.1.2 Acquisition and feedback loop

The detection circuit is schematically depicted on fig. 3.

The pick-up coil has an inductance $L$ and an internal resistance $R$. It is coupled to a capacitance, which allows to tune the whole circuit’s resonance frequency to the Larmor frequency of $^3$He. The antenna is connected to the outside of the cryostat by a coaxial cable, which has a non-negligible impedance. After that, the signal undergoes a two-stage amplification. This scheme has been preferred to a single-stage amplification to allow for both a low-noise detection and large gain versatility, given the available amplifiers.

As discussed in the introduction, the strength of radiation damping depends on the strength of the coupling between the inverted-population system and the resonant circuit. The higher the quality factor of the pick-up
circuit $Q$, the stronger the coupling and the radiation damping. In order to control this $Q$ factor, a feedback system has been implemented, re-injecting part of the signal into the pick-up circuit through a transformer of negligible inductance. Although it initially designed for negative feedback (aiming to reduce radiation damping), we use positive feedback to increase the $Q$ factor from about 7.5 to an effective factor $Q_{eff}$ of up to 60. A detailed description can be found in the Appendix.

We also sometimes needed to avoid radiation damping. For that, a Q-switch circuit was designed. It allows us to virtually ground the pick-up coil, and can be triggered by the NMR console.

The initial design featured an electronic switch, but the switching produced glitches potentially large enough to start the maser emission. It was later replaced by an electromagnetic relay, which did not create such glitches (at least on opening).

### 2.2 Sources of noise

Electromagnetic noise is a key element of our experiments, since it will be the trigger of the maser emissions, and as such, it has to be carefully estimated.

The origins of this noise can be sorted as follow: the thermal noise in the pick-up coil, the pre-amplifier noise (re-injected into the pick-up coil by the feedback loop), the noise injected of purpose and the external noise (all other sources, which can be or not be detected by the pick-up coil).
The thermal noise is the Johnson noise coming from the internal resistance of the pick-up circuit. It creates an electromotive force of power $e^2_n = 4Rk_B T$. With a temperature of 4.2K and an impedance at resonance of 2.5kΩ, it has been estimated to 0.76 nV/$\sqrt{\text{Hz}}$ [9].

The pre-amplifier noise is the electrical noise coming from the amplifiers of the feedback loop. The feedback loop contains two amplifiers, but only the first one induces a notable noise. Since the signal is already amplified when it gets through the second amplifier, its noise level is negligible. The amplifier noise can be accurately measured as being 2.5 nV/$\sqrt{\text{Hz}}$ [9].

The injected noise is the noise added in order to use different noise levels for the measurements. This injection must be done through the feedback loop, since there is no way to easily modify the configuration of the electronics of the resonant circuit, inside the cryostat. A first method for injection was the Johnson noise of a resistor added in series before the amplifier. Another method (under development) is to inject such a Johnson noise through the usually grounded input of the second-stage amplifier of the feedback loop (fig. 3).

The external noise is the one that is least known and controlled, since not all of it goes through the pick-up coil and is therefore not measured directly. It includes the field created by electric noise in the other coils of the experimental setup, as well as any other fields.

We can see that the experimental noise is dominated by the amplifier noise. Yet, the external noise has been recently estimated to be of a comparable order of magnitude (see 4.1.1). The best way to have a well-controlled noise level would then be to inject a noise strong enough to make all other noise sources negligible.

2.3 Acquisitions

2.3.1 Typical experimental procedure

The experimental procedure starts by a series of acquisition that does not require the presence of a polarized sample.

The first step in a noise measurement. It allows to check the noise level and to assess if its spectrum is white, thus targeting possible frequency leakage or other malfunction.

The second step is a $Q$-factor measurement, both with and without feedback. It is done by applying a frequency sweep through the $B_1$ coil, and measuring this signal with the pick-up circuit. The amplitude of the signal as a function of frequency is fitted to obtain the open-loop value of $Q$. In the closed-loop situation, the resonance peak height is compared to the open-loop one to obtain the $Q_{\text{eff}}/Q$ ratio.
The $^3$He is then polarized and liquefied. Small angle pulses are used to adjust the RF excitation frequency to the Larmor frequency of the spins, based on measurement of the beat frequency between the two.

The proper measurements can then start. Diffusion measurement sequences can be carried out at the beginning and the end of a series of maser emission acquisitions. The latter are repeated as long as there is enough magnetization left to be above the maser threshold.

When the magnetization has become very weak, the dipolar effects are small, a series small angle pulses can be used measure the remaining magnetization over time and monitor its decay to deduce $T_1$.

### 2.3.2 Design of the acquisition sequences

The design of the NMR sequences used to measure the statistics on maser triggering is a key element of the study: they have to allow the measurement of the total magnetization, of the noise power and of the accurate shape of the maser bursts. They also have to help saving time and magnetization along the $-z$ axis.

One of the main concerns in the design of the acquisition sequences was the fact that the magnetization flipped during a maser emission is not along the $-z$ axis any more. In the presence of distant dipolar fields, the behaviour of the magnetization is too complex to precisely know where the magnetization ended up: it is not a simple return to the $+z$ axis (fig. 4). Each maser burst also dissipates a lot of energy. These two characteristics prevent us from using sequences which would involve a succession of masers and $\pi$-pulses: in our situation, any magnetization that has left the $-z$-axis is considered to be lost.

![Figure 4: Multiple maser emission, consequence of distant dipolar effects, compared with a simple maser emission (radiation damping only).](image)

The solution is to stop the masers quickly (when the magnetization has only been tilted by a small angle $\theta$) by applying a crushing gradient, which induces a fast decay of the transverse magnetization by defocusing and diffusion. The result is a 'projection' of the magnetization on the $z$-axis.
The surviving magnetization is $M' = \cos(\theta) M$, with $\cos(\theta) \approx 1$.

The challenge is to find a good compromise between short maser acquisitions (which allow a higher repetition rate and prevent magnetization loss) and long ones (which provide a better signal-to-noise ratio). The difficulty of this task resulted in several large (if not total) maser emissions, as well as in many sequences with extremely low signal-to-noise ratio.

Knowing the exact starting time of each maser acquisition (i.e. when the system comes above maser threshold) is also essential, since it is directly related to the estimation of the starting angle. Switching the crushing gradient on and off is not a good solution, since the switching induces transient eddy-currents. The decay of these eddy-currents can last up to several seconds, and this behaviour is by no mean step-like. A better solution is to use the $Q$-switch described in 2.1.2 to drastically lower the $Q$-factor of the pick-up circuit and therefore bring the sample below maser threshold. The $Q$ damping is held for several seconds after the crushing gradient has been switched off to allow eddy-currents to decay, then it is instantaneously released by a electromagnetic relay to start the next maser emission.

The sequences also needs to allow to measure accurately the total magnetization of the sample, since the $\theta_0$ statistics depend on $M_0$. This is achieved by making a $9^\circ$-pulse before each series of maser acquisitions. The amplitude of the corresponding FID allows to compute the total magnetization of the sample, which is later corrected for each maser (see 3.2.3). This FID is then quickly crushed by a gradient in order to save acquisition time.

The total duration of each acquisition sequence is limited by the console. As a result, at the end of the sequence, the remaining magnetization is flipped by a $\pi$ pulse back along the $+z$ axis. This stable configuration allows the magnetization to wait safely until the next sequence is started.

The total acquisition sequences lasts about 1 min 20 s (fig. 5), with 2 to 4 sequences per hyperpolarized sample. This gives an average of about 20 usable masers bursts per sample.

2.3.3 Experimental configurations

The acquisitions have been performed with different sets of parameters, in order to test the reliability of the theoretical model.

A series of measurements has been performed with the lowest possible noise level (only Johnson noise from the pick-up circuit and pre-amplifier noise). This series included maser acquisition with an effective $Q$-factor $Q_{ef} \approx 29$, and with $Q_{ef} \approx 58$. Another series of 42 acquisitions has been performed after having artificially increased the noise, by inserting a large resistor in series before the pre-amplifier (fig. 4). Its Johnson noise was sent to the pick-up coil by the feedback loop and used to trigger the maser.
Figure 5: (a) Magnitude $M = \sqrt{R^2 + I^2}$ of the signal during an acquisition sequence. Initial FID of a $9^\circ$ tilting pulse during a few milliseconds, followed by 10 spontaneous maser emission, stopped by the $Q$-switching. (b) Chronogram of the NMR sequence.
After noticing discrepancies between the results at low and high noise (see 4.1.1), we made another series of high-noise acquisitions: 35 masers with the insertion of a 10 kΩ, and 47 with a 5.07 kΩ resistor. The total number of acquisitions is summarized in table 1.

<table>
<thead>
<tr>
<th></th>
<th>Low noise</th>
<th>High noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{eff}$ ≤ 29</td>
<td>165</td>
<td>124</td>
</tr>
<tr>
<td>$Q_{eff}$ ≥ 58</td>
<td>128</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Number of acquisitions made for each configuration

3 Data analysis and results

3.1 Context and goals

All data is acquired with the NTNMR software, which comes with the Tecmag-manufactured NMR hardware. The raw data sets are then exported for analysis to OriginLab’s Origin 8.0, thanks to the ONMR add-on developed for Origin 7.5.

The goal of the data analysis is to measure the shape of the maser emission (starting angle and growth rate) and the noise power, in order to compare these results to the predicted distribution of $\theta_0$.

The obstacle is the signal-to-noise ratio, which is often extremely low (SNR$<0.5$ was common). This implies that the error bars in the fitting process are going to be large. Nevertheless, the noise is expected to have a zero average - and should thus induce no systematic bias in a good fitting procedure. It is therefore important to use only linear transformations of the signal during the analysis, to maintain this property of the noise.

The signal is acquired on two channels in phase quadrature, corresponding to a real and an imaginary part. But the most needed quantity for the $\theta(t)$ fitting is the magnitude (see 1.3). The transformation from $\Re(S)$ and $\Im(S)$ to $|S| = \sqrt{\Re(S)^2 + \Im(S)^2}$ is non-linear (the noise average becomes obviously non-zero), which means the first step of the analysis is to find another way to access the magnitude of the signal.

The measurement of the noise power prior to each maser emission is performed using the beginning of each recorded maser burst, when the maser signal level is still much lower than the noise level.
3.2 Fitting of the maser burst envelope

3.2.1 Phase reconstruction

As explained, the magnitude of the maser signal has to be retrieved thanks to a linear transformation. The solution is to make a rotation in the complex plane, to bring all the signal along the real axis.

If we denote $R = \Re(S)$ and $I = \Im(S)$ the real and imaginary parts of the signal, and $M$ and $\varphi$ its magnitude and phase, we get $R + i \cdot I = M \cdot e^{i\varphi}$. Thus $(R + i \cdot I) \cdot e^{-i\varphi} = M$. This transformation is linear and should therefore preserve the vanishing average of the noise.

The difficulty is access the phase of the signal. Since the imperfect field stability and the spectral clustering effects prevent us from assuming a constant oscillation frequency, we have to use the direct measurement of the phase as $\varphi = \arctan(I/R)$. The problem is that the noise on the $R$ and $I$ channels induces a strong noise on the phase estimations. This noise can be reduced through adjacent-averaging or other smoothing methods, with risk of altering the signal shape if the smoothing is too strong.

The following method was adopted to extract a reliable (almost noiseless) phase of each maser emission.

The first step is to measure the leading frequency $f_0$ with an FFT analysis. A rotation of the signal by $-f_0 \cdot t$ is then performed as a preliminary treatment.

The processed signal is then expected to oscillate slowly enough to allow a strong smoothing (100 points Savitzky-Golay algorithm). The smoothed $R$ and $I$ are used to compute the phase of the signal. This phase is first unwrapped (to avoid $2\pi$ jumps) then strongly smoothed again (200 points Savitzky-Golay algorithm). The result is then an "average" phase, which is expected to reproduce as closely as possible what would have been the phase of the noiseless signal. The obtained signal-to-noise ratio can be very good at the end of the maser burst, but there is still a problem of noise at the beginning of the acquisition, where the original SNR is much lower than 1. It implies a phase so noisy that the unwrapping algorithm is unable to distinguish the $2\pi$ jumps from the noise.

To solve this problem, we look at the phase in the region were the SNR just starts to be decent (this doesn’t mean a specific value of SNR, but rather a region where there are no obvious unwrapping errors). The noise, even if high, should then average out. A second order polynomial fit is used on this region. The result of this fitting is then used to extrapolate the phase into the most noisy region, while the original, smoothed phase is kept for the better SNR region (fig. 6).

This extrapolation can drift quite far from the measured phase, but its role is to provide a good estimate of the phase where the signal is strong enough to affect the fitting of $M$, but too weak to allow the measurement
Figure 6: Phase reconstruction: the unwrapped phase, its smoothed version and the parabolic extrapolation of $\varphi$. Having a correct phase compensation at the very beginning of the acquisition, where the signal’s magnitude is negligible, is of little importance.

After the above-described reconstruction of the phase $\Phi$, $S = R \cdot \cos(\Phi)$ is computed and used for maser signal fitting.

### 3.2.2 Fit models

Two different models are used to fit the maser signals.

The first one is of the form $1 / \cosh((t - t_1)/T_R)$ (see the Appendix for details), as expected for a usual maser emission, without any distant dipolar field effects $[6, 7]$. This model is custom-built, and therefore suffers the major drawback of being much slower than built-in Origin models. It is used for large (or total) maser signals, and is expected to be valid up to approximately 5% below the peak maser emission - after that, the shape becomes strongly affected by distant dipolar fields (fig. 4).

This model gives an estimation of $\theta_0$, $T_R$ as well as $T_{\text{delay}}$, the expected delay between the start of the acquisition and the maximum of a maser with distant dipolar fields. If the maximum of the recorded signal is too far from the absolute maximum, degeneracy occurs for the $T_{\text{delay}}$ parameter during the fitting, which indicated that the second model has to be used instead.

This second model is a simple exponential growth model of the form $A_0 \cdot \exp((t - t_0)/T_R)$ (see the Appendix for details), where $t_0$ is the beginning
of the acquisition. It is used when the signal acquisition is stopped well before the $1/\cosh$ peak value has been reached, and is therefore undistinguishable from the exponential. This model was used for most acquisitions, as expected from the design of the NMR sequences (see 2.3.2).

Figures 7a and 7b illustrate the two situations.

Figure 7: (a) left: full maser emission, fitted up to the limit of validity by the $1/cosh$ model. (b) right: beginning of a maser emission, fitted by the exponential model.

3.2.3 Magnetization

After the value of $A_0$ has been deduced from the fitting, it has to be transformed into $\theta_0$ by comparing it to the total magnetization.

In order to do so, the initial $9^\circ$-pulse in each sequence (see 2.3.2) is used to determine the signal for the initial total magnetization $M_0$, with $M_0 \cdot \sin(9^\circ) = S(9^\circ)$ the signal level for this pulse. The magnetization for each maser is then corrected for the $T_1$ decay (measured as being 237 s in our case), as well as for the signal loss due to the crushing of the signal by the gradient at the end of each acquisition, which in the Bloch sphere corresponds to the projection of the magnetization vector from its last position onto the $-z$ axis.

If we denote by $t_n$ the starting time of the $n$-th maser, $S_{n,\text{max}}$ its maximum signal level and $M_n$ the total remaining magnetization at $t_n$, the evolution of the magnetization, accounting for $T_1$ and projection signal losses is given by:

$$M_{n+1} = M_n \cdot e^{\frac{t_n - t_{n+1}}{T_1}} \cdot \sqrt{1 - \left( \frac{S_{n,\text{max}}}{M_n} \right)^2}$$

$\theta_{0,n}$ is then simply given by $A_{0,n}/M_n$.

3.3 Noise level measurement

The last step of the data analysis is the measurement of the noise power.
This measurement cannot be done by a direct calculation of the RMS of the noise, for two reasons. First, we don’t want to include in this RMS any maser signal, as weak as it might be. Second, the signal is filtered by the pick-up circuit (even more so in the high $Q$-factor case), and therefore the noise spectrum is not flat any more (see fig. 8).

Therefore, this noise is first Fourier-transformed, and then the RMS measurement (i.e. standard deviation calculation) is done on a part of the spectrum, which has to be symmetrical, far enough from $f = 0$ to avoid the maser signal, and close enough to $f = 0$ so that the filtering effects are still negligible. The chosen frequency interval is $\left[-350 \text{Hz}, -50 \text{Hz}\right] \cup \left[50 \text{Hz}, 350 \text{Hz}\right]$.

3.4 Results

The large sample of maser emissions extracted from this series of experiments provides us with a statistical view of this phenomenon, i.e. a view of the distribution of $\theta_0$ and $T_R$ in various configurations of noise level, $Q$-factor and total magnetization.

Although many acquisitions had to be left aside due to a too low SNR or a magnetization so weak that the sample was below maser threshold, the study still provides enough data to allow us to talk about statistics (see table 2).

As described in 1.3, $x_0$ is a complex number expected to follow 2-dimensional Gaussian distribution. However, only the modulus $\theta_0$ of the maser ’complex starting angle’ was accessible. The phase should have been either measured directly, or extrapolated from the fitting of the oscillating
Table 2: Number of successfully processed maser emissions for each configuration

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Low noise</th>
<th>High noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{eff}$ ≈ 29</td>
<td>97</td>
<td>88</td>
</tr>
<tr>
<td>$Q_{eff}$ ≈ 58</td>
<td>88</td>
<td>0</td>
</tr>
</tbody>
</table>

pattern of the maser. The first could obviously not be achieved because of the signal-to-noise ratio it would require, while the second was made impossible by the field instability and the spectral clustering effects, as seen in the phase extrapolations in [3.2.1].

4 Discussion and prospects

4.1 Discussion

4.1.1 Experiments vs. Theory

The first series of results (for low noise and low $Q$-factor) looked extremely encouraging, with a very good agreement between the theory and the observed masers (fig. [9]). The average value of $\theta_0$ was correct within 7%. Their high-$Q$ counterparts looked also rather good (12% error on the average $\theta_0$), so that the study was thought to be heading towards success (fig. [10]).

Figure 9: Histogram of the measured $\theta_0$ (here noted $\theta$), normalized by the expected Gaussian standard deviation $\sigma$ derived from noise power, compared with the predicted probability density function, for low noise and low $Q$-factor.

When a few results for the high-noise configuration were analyzed, things started looking a bit more suspicious than previously thought: the experimental distribution gave us a $\theta_0$ smaller than the prediction by about a factor 2.
It was decided to make more measurements in the high noise configuration, to confirm that this surprising result was not the effect of an experimental mistake or a statistic bias. These extra measurements confirmed the noticed difference between theory and practice (fig. 11).

On the other hand, the performed measurements allowed to measure the radiation damping rate as a function of the magnetization. The expected relation (e.g. [6]) is:

$$\tau_R = \left(\frac{2\pi\eta M_0 Q_{eff} \gamma}{\eta}\right)^{-1}$$

where $\eta$ is the cavity filling factor. In our case, we measured $T_R = 1/\tau_R - 1/T_2$, so that the relation should be shifted by a constant. The experimental results can be seen of fig. 12.
Figure 12: $R_0 = 1/T_R$ as a function of the signal measured by the *Tecmag* unit, proportional to $M_0 Q_{eff}$

The points fall almost on a single line, although we would prefer this line to lie within the (tiny) error bars. The fitting to the predicted function and analysis of its results, both globally and experiment by experiment, has not been done yet, and shall be one of the improvements that can be brought to this study.

Figure 13: Average normalised $\theta_0$ as a function of the noise power (in *Tecmag* unit), shown for the different studied experimental configurations. The dashed line gives the expected value.
4.1.2 Experimental weaknesses

After seeing the very good agreement between the theory and the first experimental results, the first hypothesis to explain the discrepancy with the experiments with injection of extra noise was an experimental problem. We suspected an error in noise level estimation, given that the electronics where modified for the noise injection.

To check this hypothesis, two other methods of noise injection have been implemented, and are currently being tested. The first implies injecting the noise into the feedback loop through one of the differential amplifiers it contains (fig. 3), the other uses noise injection directly on the $B_1$ coil. Since it is orthogonal to the pick-up coil, it allows to use very high noise level while keeping a good SNR in the acquisition (fig. 1).

The preliminary results of the study with these alternative noise-injection methods are so far rather consistent with the ones presented in fig. 11.

Another source of error could be the data analysis process.

The data obtained for high noise levels have mostly a very low signal-to-noise ratio. This is a situation in which fitting error bars become larger, and it might be that fitting biases occurred. An element is disturbing when one has an intuitive look at the data, namely the shape of the fitted exponentials for masers with SNR $< 1$. The fitted curves seem to be steeper than expected, which implies both a lower $T_R$ and a lower $\theta_0$. Fig. 14 illustrate this situation: both curves offer a fitting of the data that at first glance is very convincing, yet the corresponding values of $\theta_0$ differ by a factor 3. Such a bias could even be enough to explain the average shift, yet it does not justify the total lack of higher values of $\theta_0$ among the masers with a better signal-to-noise ratio. Besides, the first results of the high-SNR experiment with noise injection through the $B_1$ coil confirm this shift of the distribution.

This bias is also rather consistent with the large error bars provided by the fit (e.g. 44% for the signal on fig. 14). The current problem of the analysis is that we have not yet found a good way to exploit them. We know that these error bars should not be symmetrical (as it can be obviously seen when the value of error bar is larger than the value of $\theta_0$ itself). Large error bars, if taken into account, could significantly change the probability distribution function, which is for now only approximated by histograms.

Another element to take into account is our lack of knowledge of the "external" noise, as defined in 2.2. Recent estimations have shown that the noise created by the $B_1$ coil is of the same order of magnitude than the noise level measured in the "low noise" configuration (to be confirmed). This would mean that actually the low-noise results, which were thought to be the best, are actually wrong. The best way to get rid of the problem of "external" noise is to make it negligible. Therefore, it is actually the high-noise configurations that should provide the best results. The $B_1$ noise injection experiment is
currently probing maser emissions with an injected noise power over ten times higher than the thermal noise and amplifier noise. The good SNR already allows us to see the effects like the Brownian motion of the spin in the begin of the maser, which was hidden by noise in previous experiments.

4.1.3 Theoretical weaknesses

The other interpretation of the persistent discrepancy between the measurements at high noise levels and theory is that the theory is wrong. We have set great store by the accuracy of the derivation, and especially the involved numerical factors. Several cross-checks later, we are convinced that the error is not a simple factor mistake.

The main weakness of the theory is that it does not take into account two major effects: distant dipolar fields, and field inhomogeneity inducing a non-exponential decay of the transverse magnetization. And yet, we know that the two effects are present and play a role in our experiments.

Tackling analytically the effect of dipolar fields makes the calculations highly non-local and is currently thought to be unmanageable [1]. But there might be a chance to efficiently include field inhomogeneities in our model. Although we cannot obtain a map of the inhomogenous field to treat the problem locally (thus neglecting dipolar fields again), we are currently trying to include the spectral distribution of the spins inside the sample (accessible via the shape of the Free Induction Decay function), in order to obtain a spectrum-dependent theory.

Simulations run thanks to the Champdip code developed by the team [10] have already shown that $T_2^*$ and dipolar fields can produce a shift of $\theta_0$ to values lower than the ones expected in the simple situation without
non-exponential decay and distant dipolar fields.

4.2 Prospects

As more experimental methods of noise injection are being tested, we can see that the flaw of our study most probably lies within the theoretical description.

We expect to be able to obtain more data with various noise levels and still a good signal to noise ratio thanks to the latest $B_1$ noise injection method. This quality of data should help reducing the uncertainties and biases resulting from the data analysis. The first obtained data have such a good SNR that it will probably even allow us to measure the starting phase of the masers, to confirm that $x_0$ obeys a symmetrical 2D Gaussian distribution.

As far as data analysis in concerned, some methodological work still needs to be done to find a way to exploit the obtained error bars. Simulations will also be run on model signals to assess for the bias mentioned in fig. [14]. On the other hand, these two problems should have their importance lessened if we use new data with much higher signal-to-noise ratio.

Most of the improvements are expected from the theoretical description. The simulation code has already been shown to be very consistent by modelling many other NMR problems in the past [11, 3, 12]. It also allows to isolate different effects (radiation damping, dipolar fields, inhomogeneous field map...) to study them one by one. It will allow us to have an idea of what needs to be modified in the theoretical description to match these various effects.

It is expected that before the end of my internship, we will have more experimental as well as simulation results, and will have implemented a theoretical description of the maser emission that would include the shape of the FID into the calculations.

This will probably be a large (if not last) step towards a complete knowledge of the dynamics of noise-triggering of NMR masers in hyperpolarized helium.

Conclusion

As discussed in this report, the study of noise-triggered maser emissions is unfinished, but already on good tracks.

The first theoretical model has proven to be too simple, but we have good expectations about the improved model currently under development. The combination of new measurements with a study of simulation results will try to confirm this new model or help further improving it.
Provided that this remaining work will be successful, the results could be published later this year.

On a broader scale, having a good understanding of the maser triggering in liquid \(^3\)He will help studying the maser as a whole: our work focuses only on the very beginning of the maser burst, but all the rest of the burst is still to be studied, where the distant dipolar fields and the instabilities they create play an important role.

References


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Appendix

A.1 Theory: detailed calculations

We start from the Bloch equations:

\[
\begin{align*}
\frac{dM_x(t)}{dt} &= \gamma (M_y(t) B_z(t) - M_z(t) B_y(t)) - \frac{M_x(t)}{T_2} \\
\frac{dM_y(t)}{dt} &= \gamma (M_z(t) B_x(t) - M_x(t) B_z(t)) - \frac{M_y(t)}{T_2} \\
\frac{dM_z(t)}{dt} &= \gamma (M_x(t) B_y(t) - M_y(t) B_x(t)) - \frac{M_z(t) - M_0}{T_1}
\end{align*}
\]

We make the small angle approximation, where \( M_z = -M_0 \). We take a field \( \vec{B}(t) = \vec{B}_0 + \vec{B}_n(t) \), where \( \vec{B}_n(t) \) is the noise field, only along the \( x \)-axis. We also add the radiation damping term in \( 1/\tau_R \).

The equations become:

\[
\begin{align*}
\frac{dM_x(t)}{dt} &= \gamma (M_y(t) B_0) - \frac{M_x(t)}{T_2} + \frac{M_x(t)}{\tau_R} \\
\frac{dM_y(t)}{dt} &= \gamma (-M_0 B_n(t) - M_x(t) B_0) - \frac{M_y(t)}{T_2} + \frac{M_y(t)}{\tau_R}
\end{align*}
\]

We set \( T_R = \frac{T_1 \tau_R}{T_2 - \tau_R} \) and define \( M_T(t) = M_x(t) + i M_y(t) \).

We also set \( \Omega_0 = \gamma B_0 \) the Larmor frequency, and go to the rotating frame. The equations become:

\[
\partial_t M_T = \frac{M_T}{T_R} + \gamma M_0 \sin(\Omega_0 t) B_n(t) + i \cos(\Omega_0 t) B_n(t)
\]

We assume \( M_T(0) = 0 \) for a perfect \( \pi \)-pulse. \( M_T(t) \) can be formally integrated to:

\[
M_T(t) = i \gamma M_0 e^{t/T_R} \int_0^t B_n(t') e^{-i \Omega_0 t'} e^{-t'/T_R} dt'
\]

The (noisy) integrand of \( K(t) \) is exponentially damped over time, so that \( K(t) \) is expected to converge towards a finite value \( K_0 \). In this limit, we have \( M_T(t) \sim i \gamma M_0 K_0 e^{t/T_R} \).

**Time-domain derivation** We suppose that the noise \( B_n(t) \) is white, therefore \( \langle B_n(t) \rangle = 0 \) and \( \langle B_n(t) B_n(t') \rangle = g^2 \delta(t - t') \).
By linearity of the integration, we can write
\[
\langle K(t) \rangle = \int_0^t \langle B_n(t') \rangle e^{-i\Omega_0 t'} e^{-t'/T_R} dt'
\]
\[
= 0
\]
\[
\langle |K(t)|^2 \rangle = \int_0^t \int_0^t \langle B_n(t') B_n(t'') \rangle e^{-i\Omega_0(t'-t'')} e^{-(t'+t'')/T_R} dt' dt''
\]
\[
= \int_0^t g^2 e^{-2t'/T_R} dt'
\]
\[
= g^2 \frac{T_R}{2} \left( 1 - e^{-2t/T_R} \right)
\]

After the damping of the noise, we therefore have
\[
\langle |K_0|^2 \rangle = g^2 \frac{T_R}{2}
\]

However, this approach might not be reliable. The assumption of the \(\delta\)-correlation of the noise in time-domain may be incorrect due to the correlations arising from the filtering of the noise by the pick-up circuit. We use a frequency approach instead, for the same reasons as during experimental measurements of the noise power.

**Frequency-domain derivation** We define the Fourier Transform of the noisy field, \(\tilde{b}(\omega)\) :
\[
B_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \tilde{b}(\omega) d\omega
\]
and rewrite \(K(t)\) as
\[
K(t) = \frac{1}{2\pi} \int_0^t dt' \int_{-\infty}^{+\infty} d\omega \tilde{b}(\omega)e^{-i\omega t'} e^{-i\Omega_0 t'} e^{-t'/T_R}
\]

Integration over time leads to :
\[
K(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{b}(\omega)e^{-t/T_R} e^{i\Psi(\omega)}}{\sqrt{(\Omega_0 + \omega)^2 + 1/T_R^2}} \sqrt{1 + e^{2t/T_R} - 2 \cos((\Omega_0 + \omega)t)e^{t/T_R}} d\omega
\]
where \(\Psi(\omega)\) is the phase of the time integral.

For a white noise, we expect \(\tilde{b}(\omega)\) to be a flat distribution with random phases, such that \(\langle \tilde{b}(\omega) \rangle = 0\) and \(\langle \tilde{b}(\omega) \tilde{b}(\omega')^* \rangle = \alpha^2 \delta(\omega - \omega')\). Actually, the white noise is filtered by the pick-up circuit, which results in a bell-shaped spectral distribution \(\alpha^2(\omega)\), centered on \(\Omega_0\) and with a bandwidth \(\sim \Omega_0/Q\).

\(K(t)\) behaves as expected at long times, converging towards a finite (complex) value : 
\[
K_0 = \frac{1}{2\pi} \int \tilde{b}(\omega) \frac{e^{i\Psi(\omega)}d\omega}{\sqrt{(\Omega_0 + \omega)^2 + 1/T_R^2}}
\]
We also obtain that \(\langle K_0 \rangle = 0\).
\[ \langle |K_0|^2 \rangle = \frac{1}{4\pi^2} \int \int \frac{\langle \tilde{b}(\omega)\tilde{b}(\omega')^* \rangle e^{i(\Psi(\omega)-\Psi(\omega'))} d\omega d\omega'}{\sqrt{\Omega_0 + \omega^2 + 1/T_R^2} \sqrt{(\Omega_0 + \omega')^2 + 1/T_R^2}} \]

\[ = \frac{1}{4\pi^2} \int \int \frac{\alpha^2(\omega)\delta(\omega - \omega') e^{i(\Psi(\omega)-\Psi(\omega'))} d\omega d\omega'}{\sqrt{(\Omega_0 + \omega)^2 + 1/T_R^2} \sqrt{(\Omega_0 + \omega')^2 + 1/T_R^2}} \]

Since \( 1/T_R \) is of the order of \( 1\text{Hz} \ll \Omega_0/Q \), we can assume \( \alpha \) to be constant over the integration, and we get:

\[ \langle |K_0|^2 \rangle \approx \frac{\alpha^2(\Omega_0)}{4\pi^2} \frac{\pi T_R}{2} = \frac{\alpha^2(\Omega_0) T_R}{2} \]

**A.2 Electronics : effect of active feedback**

**Open loop** If no active feedback is on, the tank circuit only feels the electromotive force due to the thermal (Johnson) noise \( E_J \) in the resistor. The intensity in the circuit (and therefore in the coil) is then:

\[ i_{OL} = \frac{E_J}{Z} = \frac{E_J}{R + j\omega L + \frac{1}{j\omega C}} \]

The measured signal at the output of the amplifier is:

\[ S_{OL} = G \left( \frac{i_{OL}}{j\omega C} + E_A \right) = G \left( \frac{E_J}{1 + j\omega RC - \omega^2 LC} + E_A \right) \]

where \( G \) is the amplification ratio and \( E_A \) is the amplifier’s noise.

**Closed loop** The feedback loop adds an electromotive force \( E_{CL} \) which is proportional to the output of the amplifier by a factor of \( K(\omega) \) which includes the effects of the control box and the transformer. We consider that the impedance of the tank circuit is left unchanged otherwise. The intensity in the circuit is:

\[ i_{CL} = \frac{1}{Z} \left( E_J + K(\omega)G \left( \frac{i_{CL}}{j\omega C} + E_A \right) \right) \]

\[ \Leftrightarrow i_{CL} = \frac{E_J + K(\omega)GE_A}{Z - K(\omega)G} \]

The measured signal is then:

\[ S_{CL} = G \left( \frac{i_{CL}}{j\omega C} + E_A \right) = G \left( \frac{E_J + K(\omega)GE_A}{1 + j\omega RC - \omega^2 LC - K(\omega)G} + E_A \right) \]
Q-factor The usual Q-factor is defined as usual for the open-loop circuit by $Q = 1/R\sqrt{L/C}$.

The effective Q-factor in the presence of the feedback loop $Q_{eff}$ is defined as

$$\frac{Q_{eff}(\omega)}{Q} = \frac{S_{CL}}{S_{OL}}$$

in a noiseless case.

$$S_{OL} = G \frac{i\omega}{j\omega C} = G \frac{U}{j\omega CZ}$$

$$S_{CL} = G \frac{U}{j\omega C\left(Z + \frac{K(\omega)G}{j\omega C}\right)} = G \frac{U}{j\omega CZ \left(1 + \frac{K(\omega)G}{j\omega CZ}\right)}$$

Therefore

$$\frac{Q_{eff}(\omega)}{Q} = \frac{1}{1 + \frac{K(\omega)G}{j\omega CZ}}$$

Reciprocally,

$$K(\omega)G = j\omega CZ \left(\frac{Q}{Q_{eff}(\omega)} - 1\right)$$

which at resonance yields:

$$j\omega_0 CR = j \frac{1}{Q} \Rightarrow K(\omega_0)G = j \left(\frac{1}{Q} - \frac{1}{Q_{eff}}\right)$$

A.3 Data analysis : fit models

Hyperbolic model

The model based on the shape of simple maser emissions uses a fitting function defined as:

$$y = y_0 + A \cdot \cosh\left(\frac{t_0 - T_{delay}}{T_R}\right)$$

The fixed parameters are:

- $y_0 = 0$, the offset
- $t_0$, the starting time

The fitted parameters are:

- $A$, the amplitude
- $T_R$, the characteristic growth time
- $T_{delay}$, the time after $t_0$ needed to reach the peak of $1/\cosh$

The intuitive definition of the amplitude would have been the peak height of the $1/\cosh$ function. Here, this height is given by $A \cdot \cosh\left(\frac{t_0 - T_{delay}}{T_R}\right)$, because the definition of $A$ is chosen to match the one in the exponential model.
Exponential model

The fitting function for the exponential model is defined as:

\[ y = y_0 + A \cdot e^{\frac{t-t_0}{T_R}} \]

The fixed parameters are:

- \( y_0 = 0 \), the offset
- \( t_0 \), the starting time

The fitted parameters are:

- \( A \), the amplitude
- \( T_R \), the characteristic growth time

The coherent definition of the parameters in both models allows to mix the results of both cases into the same statistics.